MATH 245 F20, Exam 2 Solutions

1. Freebie.

2. Prove that $\forall n \in \mathbb{Z}$, we must have $\frac{(n+1)(n-2)}{2} \in \mathbb{Z}$.

   There are two cases, by Corollary 1.8 (alternatively, one could cite the Division Algorithm Theorem): n is even or odd.

   Case n is even: There is some $k \in \mathbb{Z}$ with $n = 2k$. Now $\frac{(n+1)(n-2)}{2} = \frac{(n+1)(2k-2)}{2} = (n+1)(k-1) \in \mathbb{Z}$.

   Case n is odd: There is some $t \in \mathbb{Z}$ with $n = 2t+1$. Now $\frac{(n+1)(n-2)}{2} = \frac{(2t+2)(n-2)}{2} = (t+1)(n-2) \in \mathbb{Z}$.

   In both cases, $\frac{(n+1)(n-2)}{2} \in \mathbb{Z}$.

3. Let $x \in \mathbb{R}$. Prove that TFAE: (a) x is rational; (b) $7x$ is rational; (c) $x+1$ is rational.

   This problem is all about proof structures and the definition of rational. There must be at least three parts to the solution, although other TFAE proof structures are possible.

   Part $a \to b$: Assume that x is rational. Then $\exists a, b \in \mathbb{Z}$ with $b \neq 0$ and $x = \frac{a}{b}$. Now $7x = \frac{7a}{b}$. Since $7a, b \in \mathbb{Z}$ and $b \neq 0$, $7x$ is rational.

   Part $b \to c$: Assume that $7x$ is rational. Then $\exists a, b \in \mathbb{Z}$ with $b \neq 0$ and $7x = \frac{a}{b}$. Now $x = \frac{a}{7b}$ and $x+1 = \frac{a}{7b} + 1 = \frac{a+7b}{7b}$. Since $a+7b, 7b \in \mathbb{Z}$ and $7b \neq 0$, $x+1$ is rational.

   Part $c \to a$: Assume that $x+1$ is rational. Then $\exists a, b \in \mathbb{Z}$ with $b \neq 0$ and $x+1 = \frac{a}{b}$. Now $x = \frac{a}{b} - 1 = \frac{a-b}{b}$. Since $a-b, b \in \mathbb{Z}$ and $b \neq 0$, x is rational.

4. Prove or disprove: $\forall x \in \mathbb{R}$, $\lfloor x \rfloor = -\lceil -x \rceil$.

   The statement is true. We apply the definition of ceiling to $-x$ to find that $\lceil -x \rceil - 1 < -x \leq \lceil -x \rceil$. Multiply through by -1 to get $-\lceil -x \rceil \leq x < -\lceil -x \rceil + 1$. Hence, both $-\lceil -x \rceil$ and $\lfloor x \rfloor$ are integers which satisfy $m \leq x < m+1$. However, we proved that the floor of x is unique, so $-\lceil -x \rceil = \lfloor x \rfloor$.

5. Prove that $\forall n \in \mathbb{N}, 9^n > n^3$.

   Proof by (vanilla) induction. Base case $n = 1$: $9^1 = 9$, $1^3 = 1$, and $9 > 1$.

   Inductive case: Let $n \in \mathbb{N}$ and assume that $9^n > n^3$. Multiply both sides by 9 to get $9^{n+1} = 9 \cdot 9^n > 9n^3$. Now, since $n \geq 1$, we have each of the following: $n^3 \geq n^3, 3n^3 \geq 3n^2, 3n^3 \geq 3n, 2n^3 \geq 1$. Adding these, we get $n^3 + 3n^3 + 3n^3 + 2n^3 \geq n^3 + 3n^2 + 3n + 1$. This simplifies to $9n^3 \geq (n+1)^3$. Combining with the earlier $9^{n+1} > 9n^3$, we conclude that $9^{n+1} > (n+1)^3$.

6. Prove that, for every $n \in \mathbb{N}$, the Fibonacci numbers satisfy $F_{n+3} = 2 + \sum_{i=2}^{n+1} F_i$.

Proof by (vanilla) induction. Base case $n = 1$: $F_{1+3} = F_4 = 3$ and $2 + \sum_{i=2}^{2} F_i = 2 + F_2 = 2 + 1 = 3$.

Inductive case: Let $n \in \mathbb{N}$ and assume that $F_{n+3} = 2 + \sum_{i=2}^{n+1} F_i$. We add F_{n+2} to both sides, getting $F_{n+2} + F_{n+3} = 2 + F_{n+2} + \sum_{i=2}^{n+1} F_i$. The LHS simplifies to F_{n+4}, by the Fibonacci definition since $n + 4 \geq 2$. The RHS simplifies as well, giving us $F_{n+4} = 2 + \sum_{i=2}^{n+2} F_i$.

Pick your favorite, different, real numbers b, c that are not integers, to use in the rest of the exam.

The only correct answer is $b = -\frac{1}{2}$ and $c = \pi$. Anything else is NOT your favorite.

7. Solve the recurrence with initial conditions $a_0 = b, a_1 = c$ and relation $a_n = 2a_{n-1} - a_{n-2}$ (for $n \geq 2$).

We have characteristic polynomial $r^2 - 2r + 1 = (r - 1)^2$, which has the double root 1. Hence the general solution is $a_n = S1^n + Tn1^n = S + nT$, for some constants S, T. We now determine these constants using our favorite real numbers $b = -\frac{1}{2}$ and $c = \pi$.

Since $a_0 = b$, we have $-\frac{1}{2} = a_0 = S + 0 \cdot T$, so $S = -\frac{1}{2}$. Since $a_1 = c$, we have $\pi = a_1 = S + 1 \cdot T = -\frac{1}{2} + T$, so $T = \pi + \frac{1}{2}$.

Hence the solution we seek is $a_n = -\frac{1}{2} + n(\pi + \frac{1}{2})$.

8. (i) Prove or disprove that $n^b = O(n^c)$; and (ii) Prove or disprove that $n^c = O(n^b)$.

For the values of b, c above, it turns out that $-\frac{1}{2} < \pi$. Hence (i) is true and (ii) is false. If you chose b, c with $b > c$, then these need to be reversed.

Proving (i): Let $n_0 = 1$ and $M = 1$. Provided $n \geq n_0 = 1$, we also have $n^{\pi+1/2} \geq 1$, because $\pi + 1/2 > 0$. Multiplying on both sides by the positive number $n^{-1/2}$, we get $n^\pi \geq n^{-1/2}$. Therefore $|n^b| = n^{-1/2} \leq n^\pi = M|n^c|$.

Disproving (ii): Let n_0, M be arbitrary real numbers. Choose $n = 1 + \max(n_0, M^{-1/2-\pi})$. Note that n is chosen so that $n > n_0$ and also $n > M^{-1/2-\pi}$. Hence $n^{\pi+1/2} > M$, and therefore $n^\pi > Mn^{-1/2}$. So we have $|n^c| = n^\pi > Mn^{-1/2} = M|n^b|$.

Fun facts from the exam:

The most popular choice of favorite real numbers was $b = \frac{1}{2}, c = \frac{1}{4}$, chosen by 17% of students. There was a tie for second most popular choice, between $b = \frac{1}{4}, c = \frac{1}{2}$ (the most popular choice, reversed) and $b = \frac{1}{2}, c = \frac{3}{2}$, each chosen by 11% of students. NOBODY picked $b = \frac{3}{2}, c = \frac{1}{2}$ (the other popular choice, reversed).

Only 4% of students picked at least one irrational number. Only 4% of students picked at least one negative number.